HOMEWORK 5 SOLUTIONS

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Problem 8.1

The only ambiguity in the definition of g comes from the case where $x = y$, since then $x \geq y$ and $x \leq y$ both apply. To check that g is well-defined, we need to check that g evaluates to the same thing in either case. Since $x \geq y$, we have $g(x, y) = x$. Since $x \leq y$, we have $g(x, y) = y$. Since $x = y$, we may conclude that g is well-defined.

To check that $f = g$, we'll check that they agree on every input (x, y). There are two cases to consider. If $x \geq y$ then we have

$$\begin{aligned} f(x,y) &= \frac{x+y}{2} + \frac{x-y}{2} \\ &= \frac{2x}{2} \\ &= x \\ &= g(x,y). \end{aligned}$$

If $x \leq y$ then we have

$$\begin{aligned} f(x,y) &= \frac{x+y}{2} + \frac{-(x-y)}{2} \\ &= \frac{2y}{2} \\ &= y \\ &= g(x,y). \end{aligned}$$

Therefore $f = g$.

Problem 8.2

(i) $f \circ f(x) = f(x^3) = x^9$
(ii) $f \circ g(x) = f(1-x) = (1-x)^3$
(iii) $g \circ f(x) = g(x^3) = 1 - x^3$
(iv) $g \circ g(x) = g(1-x) = x$

Note: The book uses the notation $fg(x)$ to mean $f \circ g(x)$. This isn't really standard and looks a lot more like function multiplication to me. Be wary of this in future classes!

$$\{x \in \mathbb{R} | fg(x) = gf(x)\} = \{x \in \mathbb{R} | (1-x)^3 = 1 - x^3\}$$

Since $(1-x)^3 = 1 - 3x + 3x^3 - x^3$, we need to find x such that $3x^2 - 3x = 0$. This is just $\{0, 1\}$.

Problem 8.3 (iii)

Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{R} - \mathbb{Z} \\ 0 \text{ if } x \in \mathbb{Z} \end{cases}$$

Problem 8.5

(i) and (iv) are graphs of functions because for every input, there is exactly one output. In other words, there exists exactly one dot in every column. The function in (i) is as follows:

$$f(a) = z, f(b) = y, f(c) = z, \text{ and } f(d) = x.$$

The function in (iv) is as follows:

$$f(a) = y, f(b) = z, f(c) = w, \text{ and } f(c) = x.$$

(ii) is not the graph of a function because there is no dot in the column for c, so we have a domain element for which there is no assigned function value. (iii) is not the graph of a function because there are two dots in the column for b, so we fail to have more than one output for the input b. You can think of this as the graph of a function which fails to be well-defined.

Problem 15

(i) Let $f : X \to \{0, 1\}$ be the function in question, defined by $f(x) = \chi_A(x)\chi_B(x)$. We need to show that $f(x) = \chi_{A\cap B}(x)$ for all $x \in X$. There are two cases to consider:

First suppose $x \in A \cap B$. Then we have $\chi_{A\cap B}(x) = 1$. Since $x \in A$ and $x \in B$, we have $\chi_A(x) = 1$ and $\chi_B(x) = 1$. Thus $f(x) = 1 \cdot 1 = 1$. Thus $f(x) = \chi_{A\cap B}(x)$.

Now suppose $x \notin A \cap B$. Then $\chi_{A\cap B}(x) = 0$. Since x fails to be in $A \cap B$, we know $x \notin A$ or $x \notin B$. If $x \notin A$ then $f(x) = \chi_A(x)\chi_B(x) = 0 \cdot \chi_B(x) = 0$. If $x \notin B$ then $f(x) = \chi_A(x)\chi_B(x) = \chi_A(x) \cdot 0 = 0$. Thus $f(x) = \chi_{A\cap B}(x)$.

(ii) I claim that $C = A \cup B$. To see that this is correct, we need to check that the proposed function and $\chi_{A\cup B}$ agree on every input. Let $g(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$.

If $x \in A \cup B$, then $\chi_{A\cup B}(x) = 1$. To evaluate g, we'll consider 3 subcases: If $x \in A - B$ then $g(x) = 1 + 0 - 0 = 1$. If $x \in B - A$ then $g(x) = 0 + 1 - 0 = 1$. If $x \in A \cap B$ then $g(x) = 1 + 1 - 1 = 1$. In any case, we see that when $x \in A \cup B$, $\chi_{A\cup B}(x) = g(x) = 1$.

If $x \notin A \cup B$ then $x \notin A$ and $x \notin B$. We have $\chi_{A\cup B}(x) = 0$ and $g(x) = 0 + 0 - 0 = 0$. Since the functions agree on every input, they are identical.